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Free-surface gravity flow past a submerged cylinder

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The flow past a circular cylinder moving close to a free surface at high Froude number is investigated by the method of matched asymptotic expansions. In contrast with the linearized solution in which the dimensionless depth of immersion $h = h'g/U'^2$ is kept constant, in the present analysis $h \to 0$ as $Fr \to \infty$.

The inner flow model is that of a non-separated non-linear gravity-free flow past a doublet, while the linear outer solution is that of a singularity at the free surface. At deep submergence the solution coincides with the linearized solution. At moderate immersion depths the linearized solution is still valid, provided that the depth is replaced by an effective depth, larger than the actual one. For a body close to the free surface the non-linear solution differs significantly from the linearized solution.

1. Introduction

The problem of a steady potential flow of a heavy liquid past a submerged circular cylinder is considered herein. The exact equations of flow, given here for convenience of reference, are (figure 1)

$$\mathrm{Re}\left(\overline{w}'^2\frac{dw'}{dz'}+igw'\right) = 0 \quad (y' = \eta'), \tag{1}$$

$$\psi' = 0 \quad (y' = \eta'), \tag{2}$$

$$\psi' = \text{const.} \quad (x'^2+y'^2 = a'^2), \tag{3}$$

$$w' = U'; \quad \eta' = h' \quad (x' \to -\infty), \tag{4}$$

where w' is the complex velocity, $f' = \phi'+i\psi'$ the complex potential and $z' = x'+iy'$ a complex variable.

The non-linearity of (1) has generally defied attempts to solve the problem analytically, or even numerically.

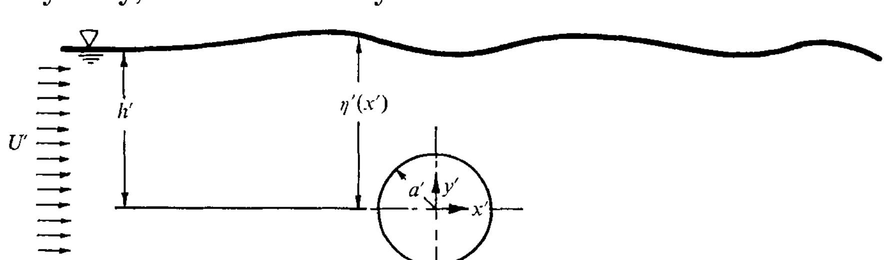

FIGURE 1. Steady free-surface flow past a submerged cylinder.

Approximate solutions have been sought by perturbation methods. The wave-making régime, in which we are interested here, corresponds to an expansion in which $\epsilon = a'g/U'^2$ is a small parameter, such that for $\epsilon = 0$, the basic unperturbed flow is uniform. The only other dimensionless parameter upon which the solution depends is $h = h'g/U'^2$. The only type of solution considered so far is that corresponding to $h = O(1)$, or to $H = h'/a'$ tending to infinity like $1/\epsilon$ as $\epsilon \to 0$.

Lamb's first-order solution of this type for a doublet, given in 1913 (Lamb 1932), may be written in the following form:

$$f = z + \epsilon^2 \left[\frac{1}{z} - \frac{1}{z-2ih} + 2i \exp(-2h - iz)\, Ei^-(2h+iz)\right], \tag{5}$$

where $$Ei^-(iu) = \int_{-\infty}^{u} \frac{e^{i\lambda}}{\lambda} d\lambda.$$

The integration is carried out in the λ lower half-plane, and the dimensionless outer variables are defined as $z = z'g/U'^2, h = h'g/U'^2$, $\epsilon = a'g/U'^2$, and $f = f'g/U'^3$.

Higher-order corrections have been derived by Havelock (1926), the procedure being given in a systematic manner by Wehausen (Wehausen & Laitone 1960, p. 574). In these higher-order approximations, only the flow in the vicinity of the body has been corrected, while the free-surface condition has been kept in its first-order version. This procedure is not consistent in principle, since the body and the free-surface contributions are of the same order. Moreover, detailed computations carried out by Tuck (1965) have shown that (i) the second-order non-linear effects become very important when the cylinder is not too far from the free surface, and (ii) the second-order correction associated with the non-linearity of the free-surface condition is more important than the one related to the body condition.

In both applications and theory it is important to determine the flow pattern in the case of submerged bodies moving close to the free surface, which is also the case in which the wave drag and lift are significant. This is the motivation of the present work, in which we let $h = h(\epsilon)$ tend to zero as $\epsilon \to 0$, i.e. we consider a limit in which the body-submergence Froude number tends to infinity as the Froude number based on the cylinder radius tends to infinity. This type of solution is, hopefully, helpful in understanding non-linear ship-wave effects, as well as other related problems.

In the usual linearized solution (5), the flow in the vicinity of the doublet ($z \to 0$), and that near the free surface, are separated. If we let $h \to 0$, the body singularity moves towards the unperturbed free surface, and the solution becomes singular there, as we can find by an additional expansion of (5), or from examination of Tuck's results. The deterioration of the solution near the origin expressed by (5) is due to the fact that the near-body and near-free-surface flows interact non-linearly even at first order.

In §2 a uniform solution for the case $h = o(1)$ is sought by the method of matched asymptotic expansions.

2. Inner and outer expansions in the auxiliary ζ plane

As will be shown later, the inner problem is that of a non-linear flow without gravity past the cylinder. To simplify the computations, we replace the cylinder by a doublet. Tuck's (1965) results suggest that the error associated with this assumption is less severe than the one connected with the non-linearity of the

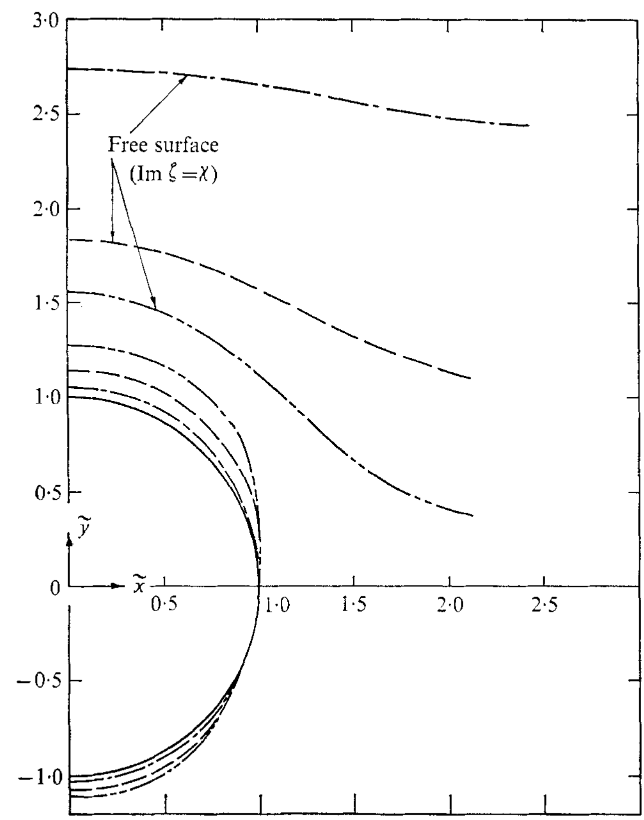

FIGURE 2. The shapes of the closed body and of the free surface for a flow past a doublet at the origin (equation (26)): ——, $\tilde{x}^2+\tilde{y}^2 = 1$; —·—·—, $\alpha/\chi = 0{\cdot}4$, $\chi = 2{\cdot}38$, $\psi_{stag} = -0{\cdot}20$; — — — —, $\alpha/\chi = 0{\cdot}7$, $\chi = 1{\cdot}28$, $\psi_{stag} = -0{\cdot}34$; —··—, $\alpha/\chi = 1{\cdot}0$, $\chi = 0{\cdot}86$, $\psi_{stag} = -0{\cdot}46$.

free-surface condition. Moreover, the present computations show that by a proper choice of the doublet strength, the shape of the resulting closed body is not far from that of a cylinder (figure 2). At any rate, the present method may be used in order to sharpen the results by adding higher order singularities to the doublet, such that the body shape could be reproduced with the desired accuracy.

To solve the non-linear inner problem it is convenient to operate in the f complex potential plane (e.g. Tulin 1963) or in an auxiliary plane related to f (e.g. Wu 1967). Let $\tilde{w}$, $\tilde{f}$, and $\tilde{z}$ be the dimensionless inner variables, referred to U'

and a' as basic velocity and length, respectively, i.e. $\tilde{w} = w'/U'$, $\tilde{f} = f'/a'U'$, $\tilde{z} = z'/a'$. The $\tilde{f}$ plane is mapped onto the auxiliary $\tilde{\zeta} = \tilde{\xi} + i\tilde{\rho}$ plane by

$$f = \tilde{\zeta} + \frac{\alpha^2}{\tilde{\zeta}} + \frac{\alpha^2}{\tilde{\zeta} - 2i\chi} - i\chi, \tag{6}$$

such that the free-surface $\tilde{\psi} = 0$ is mapped on $\tilde{\rho} = \chi$, and the doublet on $\tilde{\zeta} = 0$. The constants α and χ (strength and location of doublet in the $\tilde{\zeta}$ plane) will be determined subsequently from data from the physical plane.

The exact free-surface condition (1) becomes, in terms of the inner variables,

$$\operatorname{Re}\left[\frac{\bar{\tilde{w}}^2\tilde{w}}{1 - \dfrac{\alpha^2}{\tilde{\zeta}^2} - \dfrac{\alpha^2}{(\tilde{\zeta} - 2i\chi)^2}}\frac{d\tilde{w}}{d\tilde{\zeta}} + i\epsilon\tilde{w}\right] = 0 \quad (\tilde{\zeta} = \tilde{\xi} + i\chi). \tag{7}$$

The body condition (3) is replaced by the requirement that the closed body generated by the doublet, located at $\tilde{z} = 0$, should be as close as possible to a cylinder of unit radius. The analytical expression of this requirement is discussed in §3. The conditions at infinity (4) are replaced by matching with the outer solution.

We consider now an inner expansion,

$$\tilde{w}(\tilde{\zeta}) = \tilde{w}_0(\tilde{\zeta}) + \tilde{\delta}_1(\epsilon)\,\tilde{w}_1(\tilde{\zeta}) + \ldots, \tag{8}$$

where $\delta_1(\epsilon) = o(1)$. The expansion of (7) gives for the zero-order term (the only one considered here)

$$|\tilde{w}_0| = \text{const.} \quad (\tilde{\rho} = \chi), \tag{9}$$

i.e. a gravity-free flow. The mapping of the physical plane is found from

$$\tilde{z} = \int_0^{\tilde{\zeta}} \frac{d\tilde{\zeta}}{\tilde{w}}. \tag{10}$$

The inner solution will be shown to be singular far from the body, where gravity and inertial effects are both important. For the region far from the body ($\tilde{\zeta} \gg 1$) we consider an outer expansion. With $z = \epsilon\tilde{z}$, $f = \epsilon\tilde{f}$, $w = \tilde{w}$ (outer variables referred to U' and U'^2/g) and $\zeta = \epsilon\tilde{\zeta}$, the exact equations (1) and (4) become

$$\operatorname{Re}\left\{\frac{\bar{w}^2 w(dw/d\zeta)}{1 - \dfrac{\alpha^2\epsilon^2}{\zeta^2} - \dfrac{\alpha^2\epsilon^2}{(\zeta - 2i\epsilon\chi)^2}} + iw\right\} = 0 \quad (\rho = \epsilon\chi), \tag{11}$$

$$w = 1 \quad (\zeta \to -\infty), \quad \operatorname{Im} z = h \quad (\zeta \to -\infty + i\epsilon\chi). \tag{12}$$

The outer solution is now expanded in the ζ plane:

$$w(\zeta) = 1 + \delta_1(\epsilon)\,w_1(\zeta) + \ldots, \tag{13}$$

leading at first order, by the expansion of (11) and (12), to

$$\operatorname{Re}\left(\frac{dw_1}{d\zeta} + iw_1\right) = 0 \quad (\operatorname{Im}\zeta = 0), \tag{14}$$

$$w_1 = 0 \quad (\zeta \to -\infty), \tag{15}$$

while z is determined by a quadrature in the following form:

$$z = \int \frac{1}{w}\frac{df}{d\zeta} = \int (1-\delta_1 w_1 + \ldots)\left(1-\frac{2\epsilon\alpha^2}{\zeta}+\ldots\right)d\zeta$$

$$= \zeta + ih - i\epsilon\chi - \delta_1(\epsilon)\int_{-\infty}^{\zeta} w_1(\zeta)\,d\zeta + ec + O(\epsilon^2), \quad (16)$$

where c is an arbitrary real constant.

3. The zero-order inner solution

There are several different possible models of free-surface flow without gravity past a body. Here we are primarily interested in finding the modification of the deep submergence solution as the cylinder approaches the free surface but still remains submerged. For this reason, we assume that the free surface is continuous at zero order, and exclude the possibility of its detachment in the form of a jet. To simplify matters, (9) is replaced by

$$|\tilde{w}_0| = 1 \quad (\tilde{\rho} = \chi). \quad (17)$$

In order that the only singularity of $\tilde{w}_0$ in the lower half-plane should be of the doublet type, the function $d\tilde{z}/d\tilde{\zeta}$ has to be regular in the lower half-plane, since $\tilde{w}_0 = (d\tilde{f}/d\zeta)(d\zeta/d\tilde{z})$, and $d\tilde{f}/d\tilde{\zeta}$ (equation (6)) has the proper singularity at $\tilde{\zeta} = 0$. Hence, the function $\omega = \ln d\tilde{z}/d\tilde{\zeta} = \ln(df/d\tilde{\zeta})(1/\tilde{w}_0)$ is regular in the lower half-plane $\tilde{\rho} < \chi$, and has a known real part along $\tilde{\rho} = \chi$, given by (17) and (16):

$$\operatorname{Re}\omega = \ln\left[1 - 2\alpha^2\frac{\tilde{\xi}^2-\chi^2}{(\tilde{\xi}^2+\chi^2)^2}\right]. \quad (18)$$

With a new variable,

$$\lambda = \frac{\tilde{\zeta} - i\chi}{\chi}, \quad (19)$$

the determination of ω becomes a Dirichlet problem for the lower half-plane $\operatorname{Im}\lambda = 0$. The problem is solved by the Cauchy integral along $\operatorname{Im}\lambda = 0$, with proper care of the branch lines of ω. The function

$$1 - 2\frac{\alpha^2}{\chi^2}\frac{(\lambda^2-1)}{(\lambda^2+1)^2},$$

obtained from (18) by substituting (19) has two zeros and a double pole in the lower half-plane, located at

$$\lambda = \pm d - ie, \quad (20)$$

$$\lambda = -i, \quad (21)$$

respectively, where

$$d = \left[\frac{(2\alpha^2/\chi^2+1)^{\frac{1}{2}} + \alpha^2/\chi^2 - 1}{2}\right]^{\frac{1}{2}}, \quad (22)$$

$$e = \frac{(4\alpha^2/\chi^2 - \alpha^4/\chi^4)^{\frac{1}{2}}}{2d}. \quad (23)$$

The result of the Cauchy integral for ω is readily found as

$$\tilde{\omega} = \ln\frac{(\lambda-d-ie)^2(\lambda+d-ie)^2}{(\lambda-i)^4}, \tag{24}$$

the logarithm being taken at its principal value, such that for $\lambda \to \pm\infty$ the argument is zero.

With the aid of (19), and (24), we immediately find

$$\tilde{w}_0 = \frac{d\tilde{f}}{d\tilde{z}} = \frac{\left[1-\frac{\alpha^2}{\tilde{\zeta}^2}-\frac{\alpha^2}{(\tilde{\zeta}-2i\chi)^2}\right]\left(1-\frac{2i\chi}{\tilde{\zeta}}\right)^4}{\left[1-\chi\frac{d+i(1+e)}{\tilde{\zeta}}\right]^2\left[1+\chi\frac{d-i(1+e)}{\tilde{\zeta}}\right]^2}, \tag{25}$$

and, by further integration,

$$\tilde{z} = \tilde{\zeta}+iA\ln(\tilde{\zeta}-2i\chi)+\frac{B}{\tilde{\zeta}-2i\chi}+\frac{iC}{(\tilde{\zeta}-2i\chi)^2}+\frac{D}{(\tilde{\zeta}-2i\chi)^3}+iE, \tag{26}$$

where

$$\left.\begin{aligned} A &= 4\chi(1-e),\\ B &= 2\chi^2[d^2+3(1-e)^2],\\ C &= 2\chi^3(1-e)[d^2+(1-e)^2]^2,\\ D &= -\tfrac{1}{3}\chi^4[d^2+(1-e)^2]^2,\\ E &= -A\ln 2\chi-\frac{B}{2\chi}+\frac{C}{4\chi^2}+\frac{D}{8\chi^3}.\end{aligned}\right\} \tag{27}$$

Inspection of (25) and (26) shows that $\tilde{w}_0$ has its only singularity, of the doublet type, at $\tilde{\zeta} = 0$, while $\tilde{z}$ is regular in the half plane $\tilde{\rho} < \chi$ and vanishes at $\tilde{\zeta} = 0$. All the singularities of $\tilde{z}$ are located at $\tilde{\zeta} = 2i\chi$, the image of the doublet across $\tilde{\rho} = \chi$. The logarithmic term in (26) has zero argument along $\tilde{\xi} = 0$, $\tilde{\rho} < 2\chi$.

The next step is the derivation of the relationship between α^2 (the doublet strength) and χ (its location in the $\tilde{\zeta}$ plane). Two possibilities have been considered:

(i) The doublet strength in the $\tilde{z}$ plane is equal to unity. This is tantamount to defining a' (and ϵ accordingly) strictly as the doublet strength. The expansion of (25) and (26) in the vicinity of $\tilde{\zeta} = 0$ gives for the coefficient of $-1/\tilde{z}^2$ in the $\tilde{w}_0$ expression,

$$\frac{16\alpha^2}{[d^2+(1+e)^2]^2} = 1. \tag{28}$$

It turns out, however, that the closed body generated by the doublet, although close in its shape to a cylinder, has a radius different from unity for small χ.

(ii) The radius of the closed body is approximately equal to unity. The stagnation point of co-ordinate,

$$\tilde{\zeta}_{st} = \chi[\pm d+i(1-e)], \tag{29}$$

had been selected as the representative point of the body and its abscissa in the $\tilde{z}$ had been made equal to unity. Using (26) and (29), the following relationship is easily found:

$$|\tilde{x}_{st}| = \chi d\left\{1 - \frac{4(1-e)}{d}\arctan\frac{d}{1+e} + 2\,\frac{d^2+3(1-e)^2}{d^2+(1+e)^2} - \frac{4(1-e)\,[d^2+(1-e)^2]\,(1+e)}{[d^2+(1+e)^2]^2} - \frac{1}{3}\frac{[d^2+(1-e)^2]^2}{[d^2+(1+e)^2]^3}\,[d^3-3d(1+e)^2]\right\} = 1. \qquad (30)$$

This latter condition has been adopted in the present work in order to compute χ and α. χ and α as functions of α/χ are given in figures 3 and 4.

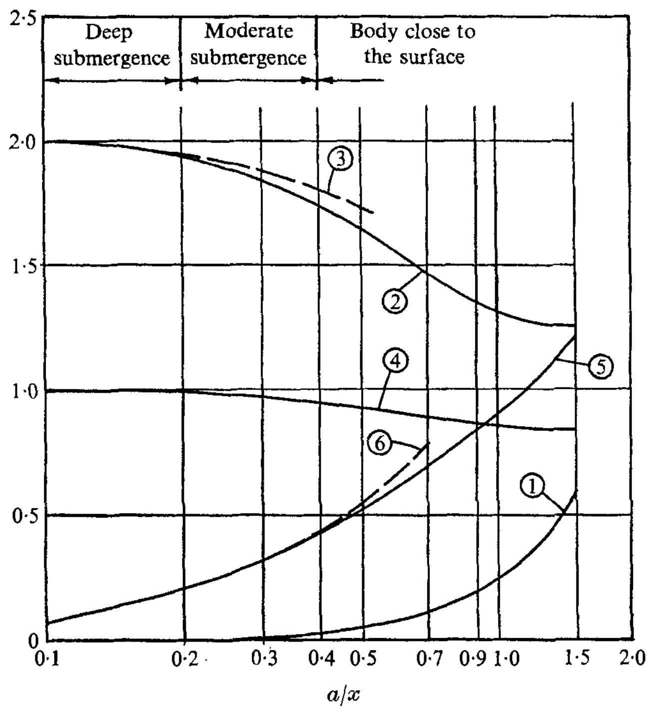

FIGURE 3. The representation of the different inner coefficients: ① A (equation (27)); ② B (27); ③ B' (36); ④ α (30); ⑤ E (27); ⑥ E'(36).

In figure 2 the shapes of the free surface and the closed body are given for a few values of α and χ. The computations have been carried out numerically, as follows:

(i) Different arbitrary values have been assigned to α/χ.

(ii) d, e, α and χ have been computed with the aid of (22), (23), and (30).

(iii) the value of

$$\psi_{st} = \text{Im}\left[\tilde{\zeta} + \frac{\alpha^2}{\tilde{\zeta}} + \frac{\alpha^2}{\tilde{\zeta} - 2i\chi} - i\chi\right] \qquad (31)$$

has been computed at the stagnation point (29). (iv) The corresponding streamlines, of $\tilde{\psi} = \tilde{\psi}_{st}$, as well as the shape of the free surface, have been computed using (31) and (26), respectively.

The streamlines are symmetrical with respect to the axis $\mathrm{Re}\,\tilde{\zeta} = 0$. This property is a result of (6), which gives

$$\mathrm{Im}\tilde{f}|_{\tilde{\zeta}=\tilde{\xi}+i\tilde{\rho}} = \mathrm{Im}\tilde{f}|_{\tilde{\zeta}=-\tilde{\xi}+i\tilde{\rho}} \tag{32}$$

and (26), which yields

$$\left.\begin{aligned}\mathrm{Re}\,\tilde{z}|_{\tilde{\zeta}=\tilde{\xi}+i\tilde{\rho}} &= -\mathrm{Re}\,\tilde{z}|_{\tilde{\zeta}=-\tilde{\xi}+i\tilde{\rho}},\\ \mathrm{Im}\,\tilde{z}|_{\tilde{\zeta}=\tilde{\xi}+i\tilde{\rho}} &= \mathrm{Im}\,\tilde{z}|_{\tilde{\zeta}=-\tilde{\xi}+i\tilde{\rho}}.\end{aligned}\right\} \tag{33}$$

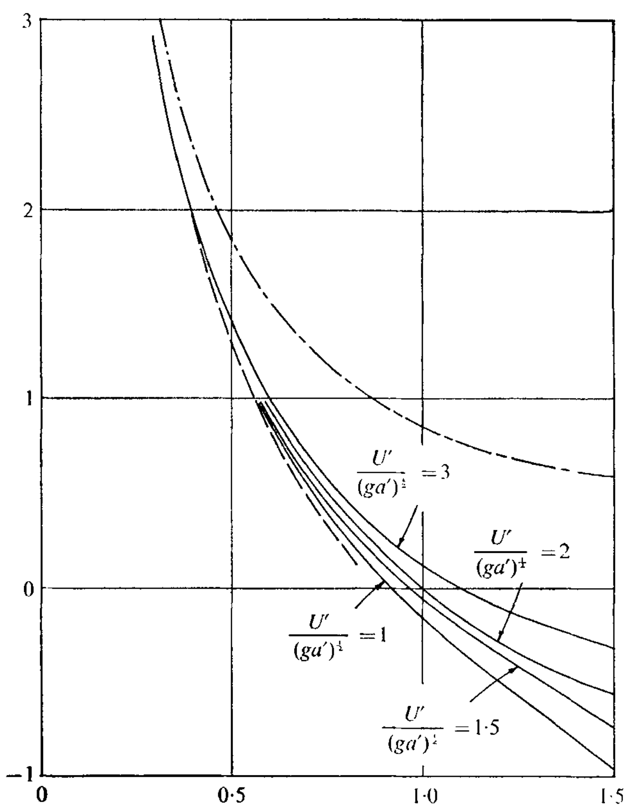

FIGURE 4. The relationship between $H = h'/a'$, χ, α/χ and $U'/(ga')^{\frac{1}{2}}$: ———, H (equation (53)); - - -, $\chi - 1/\chi$ (44); —·—·—, χ (30).

Examination of figure 2 shows that, for sufficiently small values of α/χ ($\alpha/\chi < 0{\cdot}4$), the closed body generated by the flow past the doublet practically coincides with the circular cylinder. At higher α/χ, when the body approaches the free-surface, the deviations are still not too large. This result strengthens the conclusion drawn by Tuck (1965), and suggests that representation of bodies near a free surface by singularity distributions may be quite accurate.

4. The first-order outer solution and its matching with the inner solution

We now seek $w_1(\zeta)$ satisfying (14), and (15). The behaviour of w_1 near the origin is dictated by the behaviour of the inner solution.

The expansion of $\tilde{w}_0$ (25) for large $\tilde{\zeta}$ gives

$$\tilde{w}_0 = 1-\frac{iA}{\tilde{\zeta}-2i\chi}-\frac{\alpha^2}{\tilde{\zeta}^2}-\frac{\alpha^2}{(\tilde{\zeta}-2i\chi)^2}+\frac{B}{(\tilde{\zeta}-2i\chi)^2}-\frac{A^2}{(\tilde{\zeta}-2i\chi)^2}+O\left[\frac{1}{(\tilde{\zeta}-2i\zeta)^3}\right], \tag{34}$$

while $\tilde{z}$ (26) behaves for large $\tilde{\zeta}$ like

$$\tilde{z} = \tilde{\zeta}+iA\ln(\tilde{\zeta}-2i\chi)+iE+O\left(\frac{1}{\tilde{\zeta}-2i\chi}\right). \tag{35}$$

In figure 3 α, A, B and E are given as functions of α/χ. The computations have been carried out with the aid of (27) and (30). χ as a function of α/χ is given in figure 4. Three approximate types of outer solutions are derived subsequently, corresponding to three ranges of values of α/χ and χ.

(i) *Deep submergence* ($\alpha/\chi < 0{\cdot}2$, $\chi > 5$)

In this range (figure 3) the coefficient A is negligible, while $\alpha \simeq 1$ and $B \simeq 2$. In fact, the systematic expansion of A, B, E and α ((27) and (30)) for small values of α/χ gives

$$\left.\begin{aligned} A' &= \frac{\alpha^4}{2\chi^3}+O\left(\frac{1}{\chi^5}\right),\\ B' &= 2\alpha^2+O\left(\frac{1}{\chi^2}\right),\\ E' &= -\frac{\alpha}{\chi}+O\frac{\ln\chi}{\chi^4},\\ \alpha &= 1-\frac{3}{8\chi^2}+O\frac{1}{\chi^4}.\end{aligned}\right\} \tag{36}$$

Hence, (34) and (35) degenerate in this range into

$$\left.\begin{aligned} \tilde{w}_0 &= 1-\frac{1}{\tilde{\zeta}^2}+\frac{1}{(\tilde{\zeta}-2i\chi)^2}+\ldots,\\ \tilde{z} &= \tilde{\zeta}-\frac{1}{\chi}+\ldots,\end{aligned}\right\} \tag{37}$$

or, in terms of outer variables,

$$\left.\begin{aligned} w &= 1-\frac{\epsilon^2}{\zeta^2}+\frac{\epsilon^2}{(\zeta-2i\epsilon\chi)^2}+O(\epsilon^3),\\ z &= \zeta-\frac{i\epsilon}{\chi}+O(\epsilon^2).\end{aligned}\right\} \tag{38}$$

The outer solution (13) satisfying (14) and (15), and matching w of (38) near the origin, is found immediately as follows:

$$\delta_1(\epsilon) = \epsilon^2, \tag{39}$$

$$w_1 = -\frac{1}{\zeta^2}+\frac{1}{(\zeta-2i\epsilon\chi)^2}+\frac{2i}{\zeta-2i\epsilon\chi}+2\exp(-2\epsilon\chi-i\zeta)\,Ei^-(i\zeta+2\epsilon\chi). \tag{40}$$

The matching of z ((16) and (38)) gives

$$\left.\begin{aligned} h &= \epsilon\chi, \quad \text{i.e.} \quad H = h'/a' = \chi, \\ z &= \zeta + O(\epsilon^2), \end{aligned}\right\} \tag{41}$$

where $1/\chi$ has been neglected in comparison with χ. But w_1 of (40), with z replacing ζ and h replacing $\epsilon\chi$ (equation (41)), coincides with the linearized classical solution of Lamb (5). Hence, for a sufficiently large depth of immersion ($H > 5$), the present method recovers the familiar linearized solution.

(ii) *Moderate immersion depth* ($0{\cdot}2 < \alpha/\chi < 0{\cdot}4$; $2{\cdot}5 < \chi < 5$; $2 < H < 5$) In this range A is still negligible (figure 3), $B \cong B' = 2\alpha^2$ and $E \cong E' = -1/\chi$. The outer limits of $\tilde{w}_0$ and $\tilde{z}$ are given now by the following expressions:

$$\left.\begin{aligned} w &= 1 - \frac{\epsilon^2\alpha^2}{\zeta^2} + \frac{\epsilon^2\alpha^2}{(\zeta - 2i\epsilon\chi)^2} + O(\epsilon^3), \\ z &= \zeta - \frac{i\epsilon}{\chi} + O(\epsilon^2). \end{aligned}\right\} \tag{42}$$

The outer solution $w_1(\zeta)$ is again given by (40), while (39) is replaced by

$$\delta_1(\epsilon) = \alpha^2\epsilon\chi. \tag{43}$$

Matching of z ((16) and (42)) now yields

$$h = \epsilon\left(\chi - \frac{1}{\chi}\right), \quad H = \chi - \frac{1}{\chi}, \quad \chi = \frac{H + (H^2+4)^{\frac{1}{2}}}{2}. \tag{44}$$

The wave resistance may be found from the outer solution by computing the energy radiated by the far waves. With the aid of (16), (40) and (43), the wave resistance becomes

$$R_x = \frac{R'_x}{\pi\rho g a'^2} = 4\pi\alpha^4\epsilon^2 \exp(-2\epsilon^2\chi), \tag{45}$$

while the linearized solution ((39), (40) and (41)) yields

$$R_x = 4\pi\epsilon^2 \exp(-2\epsilon H). \tag{46}$$

Comparison of (45) with (46) shows that the wave resistance based on the linearized solution is modified in the range of moderate immersion depths by two factors:

(i) The magnitude is reduced by a factor of α^4. This reduction reflects mainly the influence of the body shape, since α is the doublet strength. This factor, however, is close to unity, and its minimal value, attained for $\alpha/\chi = 0{\cdot}4$, is (figure 3) $\alpha^4 = 0{\cdot}82$.

(ii) A reduction of the magnitude of R_x by a factor $\exp[-2\epsilon(\chi - H)]$, where χ depends non-linearly on H through (44). This reduction represents the effect of the non-linearity of the free-surface condition. It states, in fact, that the effective depth of immersion χ is larger than the actual one H. As an example, for $H = 2$ and $\epsilon = 0{\cdot}5$ (i.e. $U'/(gh') = 1$), $\exp[-2\epsilon(\chi - H)] = 0{\cdot}66$. Hence, the effect of the non-linearity of the free-surface is more important in this case than that of the body. This conclusion is in qualitative agreement with the results obtained by Tuck (1965), who has computed numerically the wave resistance up to the second-order term of the linearized solution.

(iii) *Body close to the free-surface* ($\alpha/\chi > 0{\cdot}4$; $\chi < 2{\cdot}5$; $H < 2$)

In this range of the submergence parameter χ, A is no longer negligible in comparison with α and B (figure 3). The outer limits of $\tilde{w}_0$ and $\tilde{z}$ ((34) and (35)) become in this case

$$\left.\begin{aligned} w &= 1-\frac{iA\epsilon}{\zeta}+O(\epsilon^2). \\ z &= \zeta - iA\epsilon\ln\epsilon + \epsilon iA\ln\zeta + i\epsilon E + O(\epsilon^2). \end{aligned}\right\} \tag{47}$$

In the previous approximations of this section, the leading singularity in the outer limit on the inner solution ((38) and (42)) was of a doublet type. In the case of a body close to the free-surface (47), the leading singularity is of a vortex type. The vortex originates from the image of the doublet across the free surface (34); it is not, therefore, related to a circulation condition for a sharp-edged body, but to the singular behaviour of a non-linear free-surface flow without gravity past a body (Gurevich 1965).

The outer solution, satisfying (14) and (15), and matching w of (47), is found immediately by integration in the ζ plane (Wu 1967), as follows:

$$\begin{aligned} \delta_1(\epsilon) &= \epsilon, \\ w_1(\zeta) &= -A\left[\frac{i}{\zeta}+\exp(-i\zeta)\,Ei^{-}(i\zeta)\right]. \end{aligned} \tag{48}$$

Integration of (16), with the aid of (5), gives for z

$$z = \zeta + ih - i\epsilon\chi + Ai\epsilon\exp(-i\zeta)\,Ei^{-}(i\zeta) + \epsilon c + O(\epsilon^2). \tag{49}$$

In order to check that the inner and outer solutions match, $w_1(\gamma)$ and z ((48) and (49)) are expanded near $\zeta = 0$, using the expansion of $Ei^{-}(i\zeta)$:

$$Ei^{-}(i\zeta) = \gamma + \ln(-i\zeta) + \sum_{n=1}^{\infty}\frac{(i\zeta)^n}{n(n!)}, \tag{50}$$

where γ is the Euler constant ($\gamma = 0{\cdot}5772$). By substituting $\zeta = \epsilon\tilde{\zeta}$ and $z = \epsilon\tilde{z}$ in (48) and (49), and by expanding with $\epsilon \to 0$, $\tilde{\zeta} = O(1)$, we obtain

$$\left.\begin{aligned} w &= 1-\frac{Ai}{\tilde{\zeta}} - Ai\epsilon\ln\epsilon + O(\epsilon), \\ \tilde{z} &= \tilde{\zeta} + \frac{ih}{\epsilon} - i\chi + Ai\ln\tilde{\zeta} + Ai\gamma + Ai\ln\epsilon + c + O(\epsilon). \end{aligned}\right\} \tag{51}$$

The inner limit of w, (51) obviously matches the outer limit of $\tilde{w}_0$ of (47). Moreover, the order of the gauge functions in the inner expansion are $\tilde{\delta}_1(\epsilon) = \epsilon\ln\epsilon$, $\tilde{\delta}_2(\epsilon) = \epsilon, \ldots$.

The matching of $\tilde{z}$ ((51) and (47)) imposes the additional relationship,

$$h = \epsilon(\chi+E) - A\epsilon\ln\epsilon - \epsilon A\gamma, \tag{52}$$

or

$$H = \chi+E-A\ln\epsilon - A\gamma. \tag{53}$$

$H = h'/a'$, as a function of $\epsilon = a'g/U'^2$ and α/χ, is plotted in figure 4. Equations (53) and (30), and figure 4 (correspondingly) establish unique relationships between the outer parameters H, ϵ, and the inner variables α, χ. The régime of moderate submergence, and the pertinent relationship between χ and H of (44), may be easily identified on the left side of figure 4. The real constant c of (51) may be chosen equal to the difference in the arguments of $\ln\tilde{\zeta}$ of (47) and (51).

A uniform solution for w and z may be written by adding the inner and the outer solutions and subtracting their common part. For instance, from (25) and (48), we obtain for w, in terms of the outer variables,

$$w = \frac{\left[1-\frac{\alpha^2\epsilon^2}{\zeta^2}-\frac{a^2\epsilon^2}{(\zeta-2i\epsilon\chi)^2}\right]\left(1-\frac{2i\epsilon\chi}{\zeta}\right)^4}{\left[1-\chi\epsilon\frac{d+i(1+e)}{\zeta}\right]^2\left[1+\chi\epsilon\frac{d-i(i+e)}{\zeta}\right]^2} - A\epsilon\exp(-i\zeta)Ei^-(i\zeta), \tag{54}$$

which is uniform up to the term of order ϵ^2.

The solution for w and z covers a range of variation of H which is beyond the domain of validity of the linearized solution; e.g. H may be smaller than unity or even negative (figure 4), i.e. the body may protrude above the unperturbed upstream level at infinity. The physical validity of the inner model of a continuous free surface is, however, questionable for such small H. It is more probable that the free surface breaks before the body, or creates a detached jet. For this reason, the present solution is of a rather mathematical interest for small H; its physical relevance has to be determined by experiment.

The lift acting on the doublet may be found from the inner solution (25) by the Blasius theorem. Carrying out the integration yields

$$\begin{aligned} R_y &= \frac{R'_y}{\rho a' U'^2} = -\operatorname{Im}\frac{i}{2}\oint \tilde{w}_0\frac{d\tilde{f}}{d\tilde{\zeta}}d\tilde{\zeta} \\ &= -\frac{16\pi(\alpha/\chi)^4\chi}{[d^2+(1+e)^2]^2}\left\{1+4(\chi/\alpha)^2-\frac{8(1+e)[1+(\chi/\alpha)^2]}{d^2+(1+e)^2}\right. \\ &\qquad \left. -\frac{2[2d^2-10(1+e)^2]}{[d^2+(1+e)^2]^2}+\frac{12(1+e)d^2-20(1+e)^3}{[d^2+(1+e)^2]^3}\right\}. \end{aligned} \tag{55}$$

Because of the symmetry of the inner solution, there is no drag at zero order. The wave resistance may be computed from the outer solution (48), but the result will be of limited value, since we know that the next term of the outer solution, of order ϵ^2 and of doublet type, is associated with a wave comparable with those of the vortex solution (see the magnitude of A, α and B in figure 3). Hence, the computation of the wave resistance has to be based on two terms of the outer expansion at least; the development of the higher-order terms is beyond the scope of the present study.

In figure 5 the lift given by (51) is represented as function of α/χ. In the same figure, the lift computed according to the linearized solution,

$$R_{y\,\mathrm{lin}} = -2\pi\left[\frac{1}{4H^3}+\frac{\epsilon}{2H^2}+\frac{\epsilon^2}{H}-2\epsilon^3\exp(-2\epsilon H)\,\mathrm{Re}\,Ei^-(2\epsilon H)\right], \qquad (56)$$

is represented for $\epsilon = 0{\cdot}25$, by using the relationship between H and χ of (44). Inspection of figure 5 shows that, as α/χ increases, and correspondingly H decreases, $R_{y\,\mathrm{lin}}$ increases rapidly, and for $H \to 0$ it tends to infinity (56). The present non-linear theory predicts a finite lift even for H negative. The agreement between R_y of (55) and $R_{y\,\mathrm{lin}}$ of (56) is slightly improved for smaller ϵ.

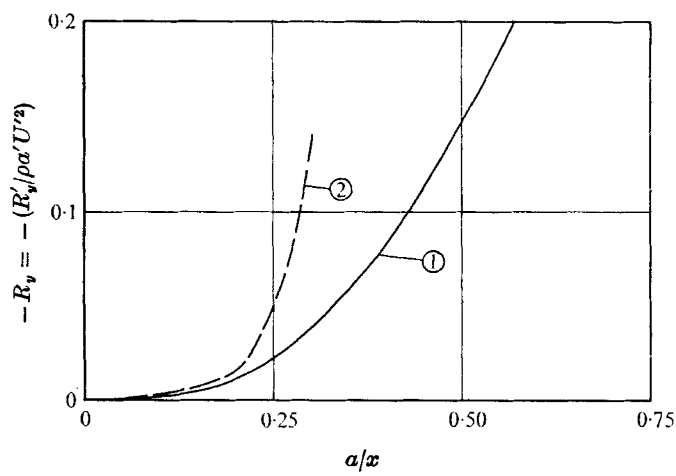

FIGURE 5. The lift force: ① $-R_y$ (equation (55)); ② $-R_{y\,\mathrm{lin}}$ (56).

5. Conclusions

The method of matched asymptotic expansions permits the determination of a uniform solution of the flow past a body close to a free-surface. The method can be extended to treat flows past bodies of different shapes, like hydrofoils. The main difficulty is the solution of the non-linear inner problem of free-surface flow without gravity past the given body. But this problem is still much simpler than the original exact non-linear problem of free-surface flow with gravity.

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